p/d operator and the unitarity property of the Fourier transform.

Assertion 4.1. If $u(z)\in \mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ and A(D) is a p/d operator with symbol analytic in Ω, then

$$F[A(D)u(z)](\zeta)=A(\zeta)\tilde{u}(\zeta).$$

Conversely, if an analytic functional $h(\zeta)$ has the form $h(\zeta) = A(\zeta)\cdot\tilde{u}(\zeta)$, where $A(\zeta)\in\mathcal{O}(\Omega)$, and $\tilde{u}(\zeta)$ is the Fourier transform of a function $u(z)\in \mathrm{Exp}_\Omega(\mathbf{C}_z^n)$, then

$$[F^{-1}h](z)=A(D)u(z).$$

This assertion makes it posible to solve the problem for p/d equations considered in Chap. 2 by the Fourier method and to again obtain the results of that chapter.

We present, for example, the analogues of Theorems 7.1 and 7.2.

THEOREM 4.1. Suppose in the Runge domain Ω the function $A(\zeta)$ is analytic and $A(\zeta) \neq 0$. Then for any function $h(z)\in \mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ there exists a unique solution $u(z)\in \mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ of the equation

$$A(D)u(z)=h(z),\quad z\in\mathbf{C}^n, \tag{4.1}$$

and u(z) is defined by the formula $u(z)=\langle \tilde{h}(\zeta), A^{-1}(\zeta)e^{z\zeta}\rangle$, where $\tilde{h}(\zeta)$ is the Fourier transform of the function h(z).

Proof. We apply the Fourier transform to Eq. (4.1). In view of the isomorphism

$$F:\mathrm{Exp}_\Omega(\mathbf{C}_z^n)\to\mathcal{O}'(\Omega)$$

in the space $\mathcal{O}'(\Omega)$ we then obtain the equivalent equation $A(\zeta)\tilde{u}(\zeta)=\tilde{h}(\zeta)$, whence $\tilde{u}(\zeta)=A^{-1}(\zeta)\tilde{h}(\zeta)$. By the inversion formula $u(z)=\langle A^{-1}(\zeta)\tilde{h}(\zeta), \exp z\zeta\rangle\equiv\langle\tilde{h}(\zeta), A^{-1}(\zeta)e^{z\zeta}\rangle$ is the solution of Eq. (4.1). This is what was required.

The dual result can be formulated similarly.

THEOREM 4.2. Let $A(\zeta)\in\mathcal{O}(\Omega)$, $A(\zeta)\neq 0$, $\zeta\in\Omega$. Then for any right side $h(z)\in \mathrm{Exp}'_{\Omega^-}(\mathbf{C}_z^n)$ there exists a unique solution of Eq. (4.1) which is defined by the formula

$$u(z)=F[A^{-1}(-\zeta)\langle h(z), e^{z\zeta}\rangle](z) \tag{4.2}$$

(we recall that $\Omega^-=\{\zeta\in\mathbf{C}^n: -\zeta\in\Omega\}$).

Proof. We use the isomorphism

$$F:\mathcal{O}(\Omega^-)\to \mathrm{Exp}'_{\Omega^-}(\mathbf{C}_z^n)$$

(the variables z and ζ have changed roles as compared with the general theory). Then in correspondence with the inversion formula we go over from Eq. (4.1) to the equivalent equation for functions in $\mathcal{O}(\Omega^-)$:

$$A(-\zeta)\hat{u}(\zeta)=\hat{h}(\zeta),$$

where $\hat{u}(\zeta)=\langle u(z), \exp z\zeta\rangle$. From this we immediately find that the desired solution has the form (4.2). There is what was required.

It is obvious that the Cauchy problem considered in Sec. 7, Chap. 2 can also be solved by the Fourier method. Reformulation of the corresponding results occasions no difficulties. The results are the same as in Sec. 7, Chap. 2.

Thus, in conclusion it can be noted that the operator method (Sec. 7, Chap. 2) and the Fourier method are equivalent within the framework of the exponential theory.

CHAPTER 3

P/D OPERATORS WITH VARIABLE ANALYTIC SYMBOLS

1. Definition of a P/D Operator with Variable Symbol

Let $A(z, \zeta)$ be an analytic function of the variables $z\in\mathbf{C}^n$ and $\zeta\in\Omega$, where $\Omega\subset\mathbf{C}_\zeta^n$ is a Runge domain. We have

$$A(z,\zeta)=\sum_{|\alpha|=0}^{\infty} z^\alpha A_\alpha(\zeta), \tag{1.1}$$

where $A_\alpha(\zeta)\in\mathcal{O}(\Omega)$.

In correspondence with this expansion we set

$$A(z, D)u(z) \overset{\text{def}}{=} \sum_{|\alpha|=0}^{\infty} z^{\alpha} A_{\alpha}(D) u(z),$$

where $A_\alpha(D)$ are p/d operators with symbols $A_\alpha(\zeta)$.

We suppose that the function $A(z, \zeta)$ is in z an entire function of minimal type. More precisely, suppose for any $\varepsilon > 0$ and any compact set $K \subset \Omega$ there exists a number $M > 0$ (depending, generally speaking, on ε and K) such that for all $z \in \mathbf{C}^n$ and $\zeta \in K$

$$|A(z, \zeta)| \leqslant M \exp \varepsilon |z|. \tag{1.2}$$

Assertion 1. If condition (1.2) is satisfied, then the mapping

$$A(z, D): \mathrm{Exp}_{\Omega}(\mathbf{C}_z^n) \to \mathrm{Exp}_{\Omega}(\mathbf{C}_z^n)$$

is defined and continuous.

Proof. Indeed, in view of the uniform convergence of the series (1.1) on any compact set $K \subset \Omega$, for any $z \in \mathbf{C}^n$ from formula (5.4) (Sec. 5, Chap. 1) giving the integral representation of a p/d operator we have

$$A(z, D)u(z) = \frac{1}{(2\pi i)^n} \sum_{\lambda} e^{\lambda z} \int_{\Gamma_{\varepsilon,\lambda}} A(z, \lambda + \zeta) B\varphi_{\lambda}(\zeta) e^{-z\zeta} d\zeta.$$

From condition (1.2) it now follows immediately that $A(D)u(z) \in \mathrm{Exp}_{\Omega}(\mathbf{C}_z^n)$.

Further if $\varphi_{\lambda\nu}(z) \to \varphi_\lambda(z)$ $(\nu \to \infty)$ in $\overline{\mathrm{Exp}}_r(\mathbf{C}_z^n)$, then $B\varphi_{\lambda\nu}(\zeta) \to B\varphi_\lambda(\zeta)$ uniformly for $|\zeta_j| \geqslant r_1 > r$ $(j = 1,\dots,n)$ and, in particular, on the contour $\Gamma_{\varepsilon,\lambda}$. From this it follows that $A(z, D) \times u_{\lambda\nu}(z) \to A(z, D)u_\lambda(z)$ in $\mathrm{Exp}_{\Omega}(\mathbf{C}_z^n)$ and hence $A(z, D)u_\nu(z) \to A(z, D)u(z)$ in $\mathrm{Exp}_{\Omega}(\mathbf{C}_z^n)$, provided that $u_\nu(z) \to u(z)$ in $\mathrm{Exp}_{\Omega}(\mathbf{C}_z^n)$. This is what was required.

Example. Let

$$A(z, \zeta) \equiv \sum_{|\alpha|=0}^{m} \mathscr{P}_{\alpha}(z) A_{\alpha}(\zeta),$$

where $\mathscr{P}_\alpha(z)$ are polynomials. Then to the symbol $A(z, \zeta)$ there corresponds a p/d operator $A(z, D)$ with polynomial coefficients.

2. The Required Spaces

In the next section we shall consider the Cauchy problem for systems of p/d equations with variable symbols. To study them we introduce the required spaces.

Let $u(z) = (u_1(z),\dots,u_N(z))$ be a vector-valued function where $u_j(z): \mathbf{C}^n \to \mathbf{C}^1$ are entire functions. Further, let $m = (m_1,\dots,m_N)$ be an integral vector with $m_j \geqslant 0$ $(j = 1,\dots,N)$; $r \geqslant 0$ is some number

Definition 2.1. We set

$$\overline{\mathrm{Exp}}_{m,r}(\mathbf{C}_z^n) = \left\{ u(z): \| u_j(z) \|_{m_j,r} \equiv \sup_{z \in \mathbf{C}^n} |u_j(z)| (1 + |z|)^{-m_j} \exp(-r|z|) < \infty,\ j = 1, \dots, N \right\}.$$

It is not hard to see that $\overline{\mathrm{Exp}}_{m,r}(\mathbf{C}_z^n)$ is a Banach space with norm

$$\| u(z) \|_{m,r} \equiv \| u_1(z) \|_{m_1,r} + \dots + \| u_N(z) \|_{m_N,r}.$$

Further, we say that $u(z) \in \overline{\mathrm{Exp}}_{m,r}(\zeta_0; C_z^n)$, where $\zeta_0 \in \mathbf{C}^n$ if $u(z)\exp(-\zeta_0 z) \in \overline{\mathrm{Exp}}_{m,r}(\mathbf{C}_z^n)$.

The spaces introduced will be the spaces of initial data for the Cauchy problem.

We now consider the spaces of the variables $t \in \mathbf{C}^1$, $z \in \mathbf{C}^n$ in which a solution of the Cauchy problem will be found.

Let $\sigma > 0$ and $\delta > 0$ be some numbers. We denote by $\mathcal{O}(\delta; \overline{\mathrm{Exp}}_{m,r+\sigma|t-t_0|}(\zeta_0; \mathbf{C}_z^n))$ the Banach space of functions $u(t, z)$ analytic in t for $|t - t_0| \leqslant \delta$, whereby $u(t, \cdot) \in \overline{\mathrm{Exp}}_{m,r+\sigma|t-t_0|}(\zeta_0; \mathbf{C}_z^n)$. We define the norm in this space by the equality

$$\| u(t, z) \|_{\delta; m, r, \sigma} \equiv \sum_{j=1}^{N} \max_{|t-t_0| \leqslant \delta} \| u_j(t, z) \exp(-\zeta_0 z) \|_{m_j, r+\sigma|t-t_0|}$$